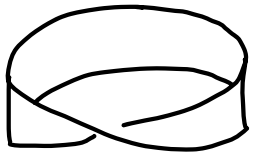
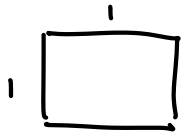
Optimal Paper Möbius Bands

Let's make a Möbius band out of paper!

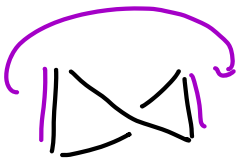


easy:





hard

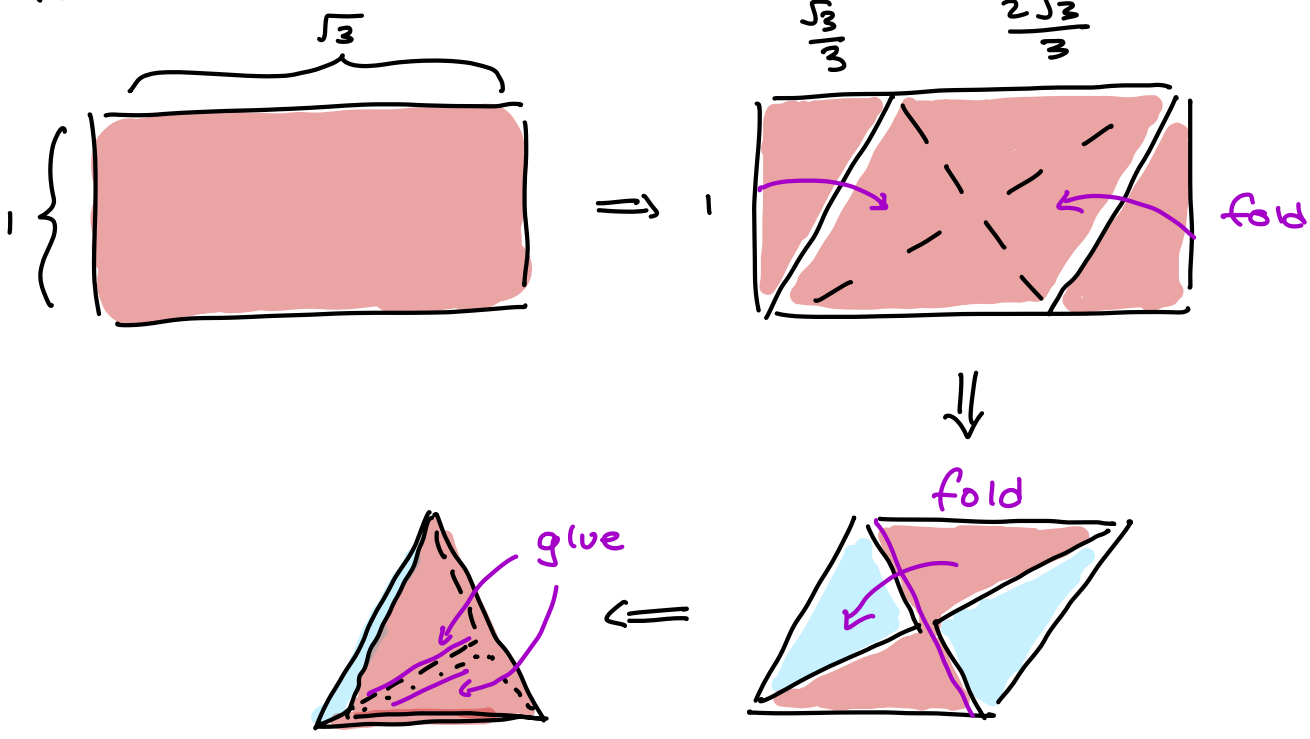


can't connect

Q Find $\inf\{\lambda : 1 \times \lambda$  makes paper möbius band$\}$

(i.e. admits isom. embedding to $\mathbb{R}^3$

One limit construction:



So $\inf \leq \sqrt{3}$.

Thm (Rich Evan Schwartz '24) (☆)

$\inf = \sqrt{3}$.

Thm (S '24)

Given (M_{λ_n}, I_n) isom. embeddings, (mobius band, λ_n)

$\lim_{n\to\infty} \lambda_n = \sqrt{3}$, we have that the I_n converge uniformly (up to isom.) to the  embedding above.

We'll prove Thm (☆)

Idea:

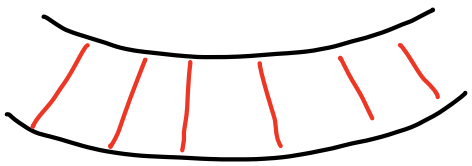
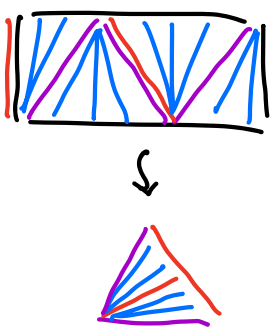
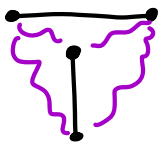
1) 

image of mobius band is foliated by straight line segments (bends)

2) ∃ Two bends which are coplanar AND ⊥. ("T pattern")

This ∃ by Borsuk Ulam



3) 

Finish by showing an equilateral triangle minimizes Mobius band length w/ "T pattern"

From now on, consider $I : M_\lambda \longrightarrow \mathbb{R}^3$ isom. embed.

We wish to show $\lambda > \sqrt{3}$.

§ 1 Bends

Def: Foliation by 1-dim submanifs: Partition which locally looks like ////

Lemma 1 $I(M_\lambda)$ admits a foliation by straight line segments (bends).

Def $I^{-1}(\text{bend}) =:$ prebend.

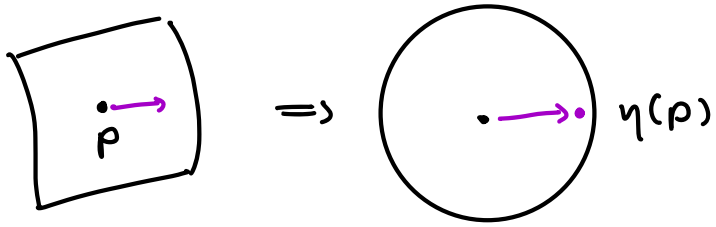
For Lemma 1, recall Gauss map

$$\eta : \Sigma \overset{\subset \mathbb{R}^3}{\longrightarrow} S^2$$

(Σ: oriented embedded surface)

$$p \longrightarrow \text{unit normal vector at } p$$

(it's ok that our M_λ is not ori: we'll only use local ori; slices of it.)



Basic DG Facts:

1) We can identify $T_{\eta(p)} S^2$ w/ $T_p \Sigma$, so we get

$$d\eta_p : T_p \Sigma \longrightarrow T_{\eta(p)} S^2 \cong T_p \Sigma$$

This map is self-adjoint, so it's diagonalizable w/ $\perp$ eigenvectors

2) $\det d\eta_p =: K(p)$ Gauss curvature

$-\frac{1}{2}\operatorname{tr} d\eta_p =: H(p)$ Mean curvature

Thm Egregium: Gauss curvature is an isometry invariant. Notably for M_λ, $K(p) \equiv 0$

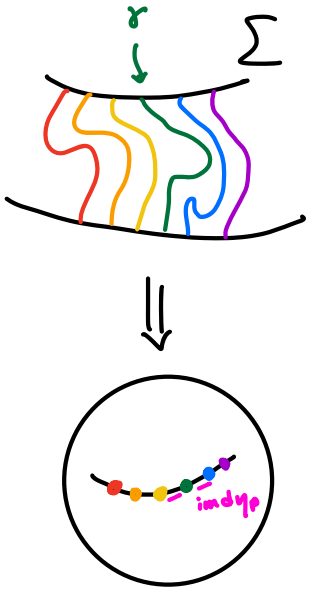
To prove Lemma 1 (sketch):

Consider $U = \{ p \in \Sigma : H(p) \neq 0 \}$.

Since $d\eta_p$ is self-adjoint and it's trace $\neq 0$, we have that $\{ \ker d\eta_p, \operatorname{im} d\eta_p, \eta \}$ form an $\perp$ basis of VFs at every pt.

(really, pick nonzero vectors in here.)

By Const. Rank Thm, since $\operatorname{rk} d\eta_p|_U \equiv 1$, η locally factors thru a map from Σ to an emb. 1-manif. Taking preimages foliates Σ by curves integral to $\ker \eta_p$.

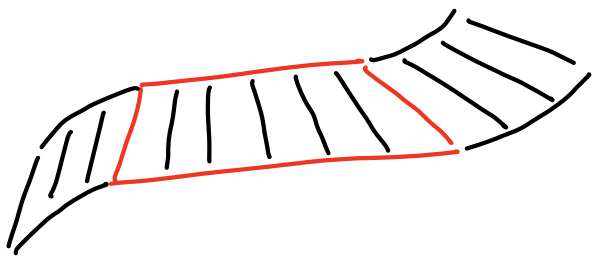


On each curve γ:

- η is const' (by integrating)
- η locally factors as a map to a 1-dim immersed submanif of S^2. So $\operatorname{im} d\eta_p$ is (loc., hence glob.) const' on γ (since it's the vector tangent to this submanif)

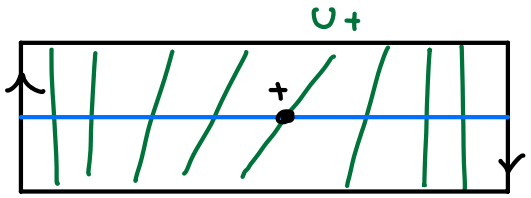
So $\ker \eta_p$ is also const on γ. But $\gamma \parallel \ker \eta_p$, so γ is a straight line segment.

This foliates U by bends. From here, not hard to show that $\Sigma \setminus U$ looks like a collection of trapezoids, in which you can easily construct a foliation. $\square$



§2 Finding a T

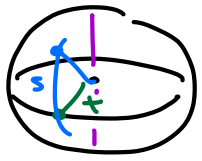
So $I(M_\lambda)$ is foliated by bends. Parameterize prebends in M_λ by where they hit the centerline $\mathbb{R}/\lambda\mathbb{Z} \times \{\frac{1}{2}\}$; specifically, let U_t be the bend hitting $(t, 1/2)$:



View pairs of distinct bands (U_t, U_s) by mapping them to

$(v_t, v_s) \longrightarrow (t, s-t) \in \mathbb{R}/\lambda\mathbb{Z} \times (0, \lambda)$

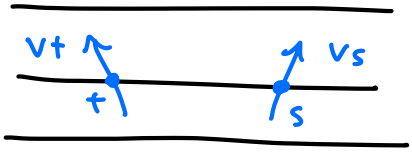
Embed $\mathbb{R}/\lambda\mathbb{Z} \times (0,\lambda) \hookrightarrow S^2$

$(t, s) \longrightarrow$ 

This "compactifies" space of bend pairs:

- north pole $= \lim_{s \to t^+} (v_t, v_s)$
- south pole $= \lim_{s \to t^-} (v_t, v_s)$

Now given (v_t, v_s), pick $v_t \parallel v_t$, $v_s \parallel v_s$, v_t, v_s unit vectors, such that moving rightwards $t \to s$ carries the v_t "orientation" to v_s.



Let m_t be the midpt of v_t. Def':

$$S^2 \xrightarrow{\varphi} \mathbb{R}^2$$

$$(v_t, v_s) \to (v_t \cdot v_s, \underbrace{(m_t - m_s) \cdot (v_t \times v_s)}_{\text{if } = 0,\ (m_t - m_s, v_t, v_s) \text{ coplanar}})$$

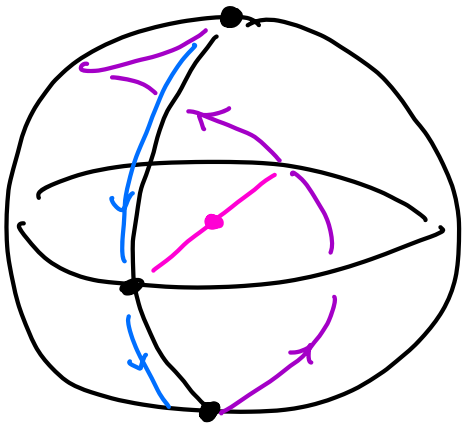
This is well def'd:

- north pole $\longrightarrow (1,0)$
- south pole $\longrightarrow (-1,0)$.

Also $\varphi^{-1}((0,0))$ consists of $\top$ patterns!

Now the map $(v_+, v_s) \to (v_s, v_+)$

- is conj' to the antipodal map on S^2
- Negates φ



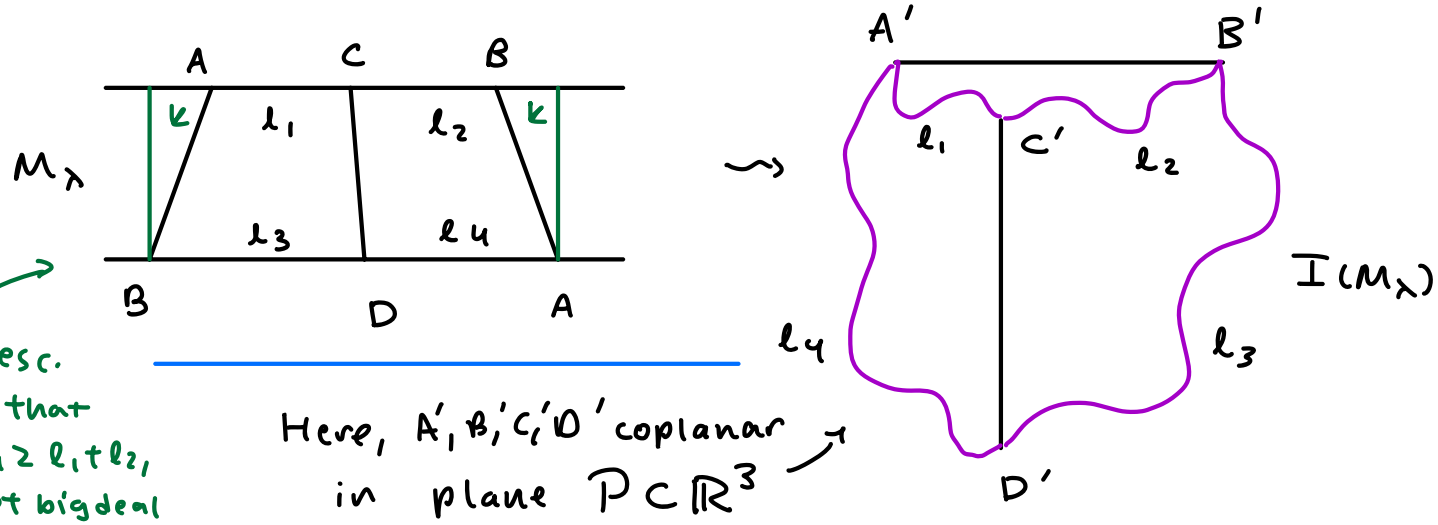
So by Borsuk-Ulam, φ has a zero; i.e. $\exists$ a T pattern.

§3 Finishing

Intuitively, easy to see why T pattern finishes; I have to interpolate btwn edges in T pattern, & this is shortest in $\triangle$ config.

Details:

Consider bends (v_+, v_s) forming a T pattern. Label the diagram as follows:



Note $C'D' \geq 1$. Also $A'B' \geq AB = \sqrt{1+k^2}$

Easy lemma:

$A'D' + D'B' \geq \sqrt{A'B'^2 + 4}$, w/ equality iff $C' \in A'B'$, $A'D' = B'D'$.

false

Ok, this gives

$$2\lambda = \ell_1 + \ell_2 + \ell_3 + \ell_4$$
$$\geq A'B' + A'D' + B'D'$$
$$\geq \sqrt{1+k^2} + \sqrt{5+k^2}$$

Also

$$2\lambda = \ell_1 + \ell_2 + \ell_3 + \ell_4$$
$$\geq 2(\ell_3 + \ell_4) - 2k$$
$$> 2\sqrt{5+k^2} - 2k$$

Combining the inequalities gives $\lambda > \sqrt{3}$.

(Not hard to show that $\lambda \longrightarrow \sqrt{3}$ gives $I(M_\lambda) \longrightarrow \triangle$).

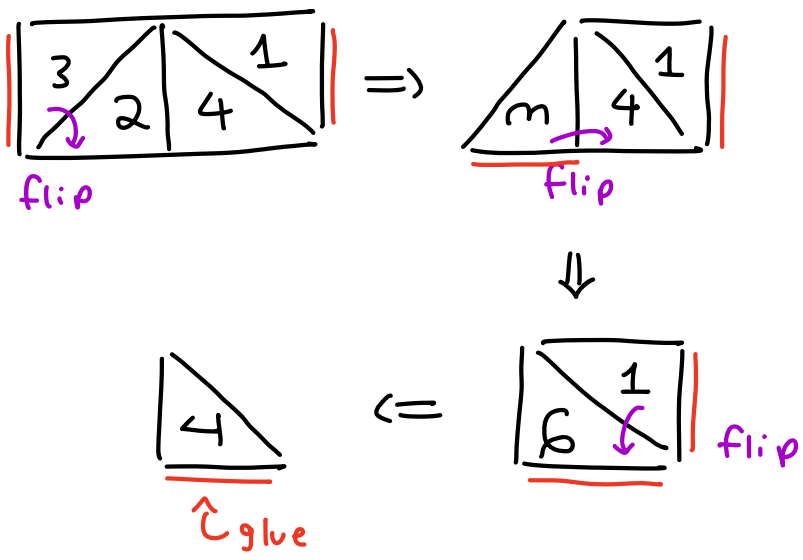
■

Next steps

What about more half twists?

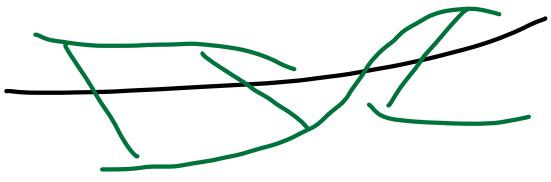
- 2 half twists?

One (but not only!) optimal construction:

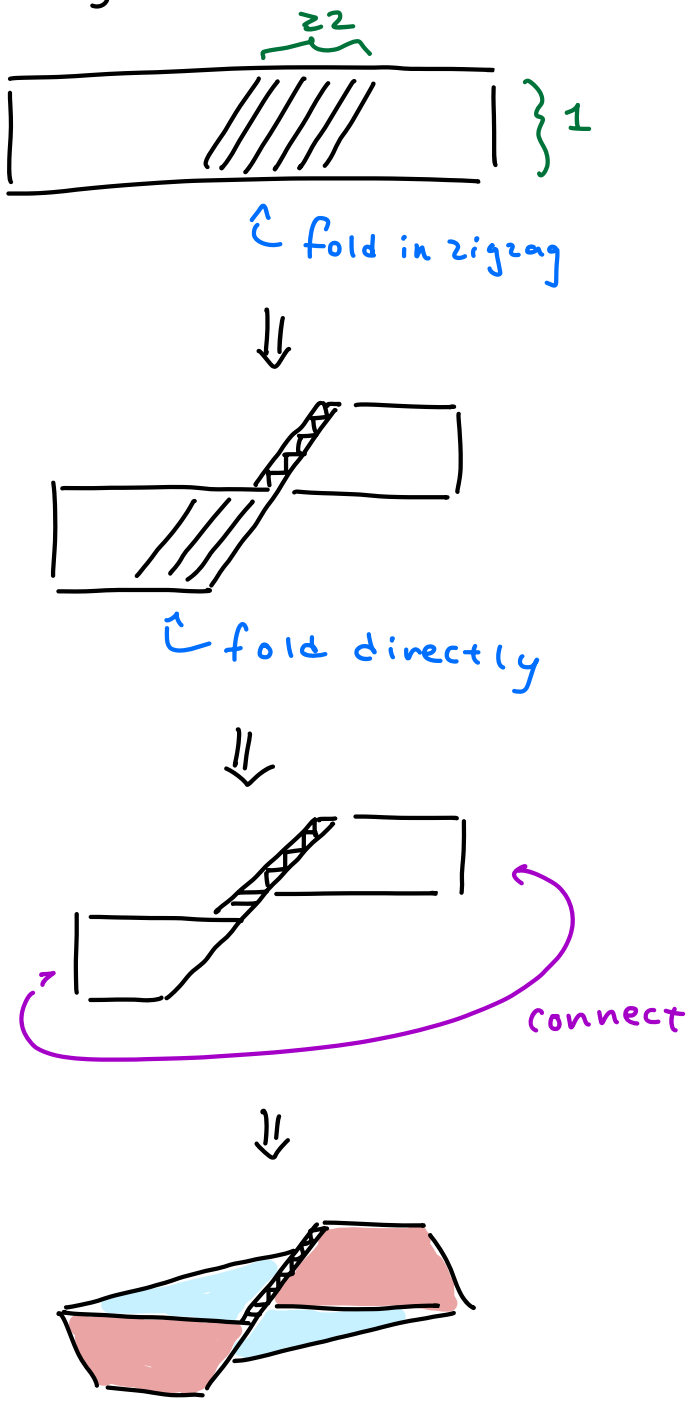


- n half twists?

  ⇝ boundary always forms $(2, n)$-cable over $1/2$ centerline



↝ For any n, $3+\sqrt{7}+\varepsilon$ works (Hennessy '24):



In fact $3\sqrt{3}+\varepsilon$ works (Neinhaus)

Thanks!