Morse Theory on Path Spaces

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0.1 Preface

This is an expository note on Bott's Morse theory proof that the second homotopy group of any connected Lie group is trivial. We follow [Mil63]; notably, while the writing of this article is original, none of its theorems or proofs are. Readers are expected to have a working knowledge of Riemannian Geometry (e.g. Math 230a, or [Mil63, Chapters 8-10]) and algebraic topology (e.g. Math 231ab). Some exposure to Lie theory (e.g. [FH91, Part II]) is also helpful. Throughout this document, we'll assume all Lie groups to be real, and assume (co)homology to have integer coefficients.

1 Introduction

The objective of this article is to prove the following theorem of Bott.

Theorem 1.1 ([Bot56])**.** *Every connected Lie group G has trivial π_2.*

By considering the universal cover of G, then applying the following theorem of Cartan, it suffices to consider compact simply connected G.

Definition 1.2. Let a *maximal compact subgroup* of a Lie group G be a compact subgroup, maximal (under inclusion) amongst all such subgroups.

Theorem 1.3 (Cartan)**.** *Every connected Lie group G deformation retracts onto any of its maximal compact subgroups.*

We will follow Bott's original proof using Morse theory. By considering the long exact sequence on homotopy groups of the fibration

$$\Omega G \to * \to G,$$

it suffices to prove that $\pi_1 \Omega G = 1$; in fact, since $H_1(\Omega G) = (\pi_1(\Omega G))_{\text{ab}} = (\pi_2(G))_{\text{ab}} = \pi_2(G)$, it suffices to prove $H_1(\Omega G) = 1$. In fact Bott proved a much stronger characterization of ΩG.

Theorem 1.4 ([Bot56])**.** *For any compact simply connected Lie group G, ΩG is homotopy equivalent to a CW complex with no odd-dimensional cells, and finitely many cells of any given even dimension.*

This implies that the (cellular) homology of ΩG is zero in every odd degree and free and finitely generated in every even degree. Notably this implies $\pi_2 G = H_1(\Omega G) = 0$, thus proving Theorem 1.1. This also implies that $\pi_3 G = \pi_2(\Omega G)$, which is equal to $H_2(\Omega G; \mathbb{Z})$ by Hurewicz, is of the form $\mathbb{Z}^t$ for some integer t.

Example 1.5. The special unitary group SU_2 consists of all the complex 2×2 matrices U of determinant 1 whose inverse is equal to their conjugate transpose. A rudimentary calculation reveals that all such matrices are of the form

$$U(\alpha, \beta) = \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}$$

where $\alpha, \beta \in \mathbb{C}$, and $|\alpha|^2 + |\beta|^2 = 1$. The map $U(\alpha, \beta) \to (\alpha, \beta) \in \mathbb{C}^2$ thus maps SU_2 diffeomorphically to S^3.

Now, from class we know that $H^*(\Omega S^3; \mathbb{Z})$ is the divided power algebra $\Gamma[x]$ with generator x of degree 2. Hence by the Universal Coefficient Theorem, $H_k(\Omega S^3; \mathbb{Z})$ is trivial for odd k and isomorphic to $\mathbb{Z}$ for even k. Thus the CW complex in Theorem 1.4 is infinite-dimensional in this case.

In fact, this is generally true for any compact simply-connected Lie group G that is not just a point. Indeed, by investigating the homology Serre spectral sequence of $\Omega G \to * \to G$, if $H_* \Omega G$ is only supported on finitely many degrees, then by considering the maximal k, l such that $H_k \Omega G$ and $H_l G$ are nontrivial, we have that the diagonal $s + t = k + l$ in the E_2 page consists only of the nonzero term $H_l(G; H_k(\Omega G))$, which only ever interacts with trivial differentials. Thus the E^∞ page cannot give the homology of a point unless $l = k = 0$, which by Hurewicz and Whitehead imply that G is homotopy equivalent to a point. But it is well-known that if G is n-dimensional then $H_n(G; \mathbb{F}_2) \cong \mathbb{F}_2$; thus $n = 0$ and G is a point as desired. $\square$

Here is a sketch of the proof of Theorem 1.4. The idea is to use *Morse theory*, which tells us that given a manifold M and a "height" function $f : M \to \mathbb{R}$ with only nondegenerate critical points (i.e. a Morse function), M admits a CW decomposition with k-cells corresponding to index k critical points of f. To apply this setup to ΩG, equip G with a bi-invariant Riemannian metric. The height function we will use on ΩG is the energy function

$$E(\gamma) = \int_0^1 \left| \frac{d\gamma}{dt} \right|^2 dt$$

where $\gamma \in G$ is a loop $\gamma : [0, 1] \to G$. The "critical points" of the energy function are precisely geodesics, and one may check via a computation that the index of each geodesic as a critical point of E is even when the ambient manifold is a compact Lie group. With some assumptions and additional results, Morse theory thus says that ΩG has a CW decomposition with only even cells (with finitely many of each dimension), as desired.

There are some details to be ironed out. First is the Morse theory setup for finite-dimensional manifolds, which we will detail in Section 2. We'll then analogize some of this setup to path spaces of Riemannian manifolds in Section 3. To apply some of the structural theorems from usual Morse theory however, we will need to explicitly construct a finite-dimensional approximation of our path spaces, which we will do in Section 4. Finally in Section 5 we'll piece together a proof of Theorem 1.4.

2 Morse Theory

Let's first quickly refresh ourselves on Morse theory. The basic idea of the theory is simple: I can understand any given manifold by ascribing to it a generic height function, then looking at how the manifold changes as I move "upwards". To make this theory rigorous, let us introduce the following notions. Throughout, fix a smooth (not necessarily compact) manifold M.

Definition 2.1. Consider a smooth map $f : M \to \mathbb{R}$ and a critical point $p \in M$ of f. Consider any two vectors $v, w \in T_pM$, and extend them both locally to vector fields $\tilde{v}, \tilde{w}$. Define the *Hessian* $H : T_pM \times T_pM \to \mathbb{R}$ by sending

$$H(v, w) := (\nabla_{\tilde{v}} \nabla_{\tilde{w}} f)(p) - (\nabla_{\tilde{w}} \nabla_{\tilde{v}} f)(p) = (\nabla_{[\tilde{v},\tilde{w}]} f)(p).$$

Here $\nabla_{\tilde{v}}$ denotes the directional (generally, covariant) derivative with respect to $\tilde{v}$.

One may check that the Hessian as described is well-defined, and is a symmetric bilinear form on T_pM. Note that in local coordinates $x_1, \dots x_n$, H is given by

$$H = \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{i,j}$$

Definition 2.2. A critical point of $f : M \to \mathbb{R}$ is *non-degenerate* if H is, and *degenerate* otherwise.

Definition 2.3. A *Morse function on M* is a smooth function whose critical points are all non-degenerate.

Using Sard's Theorem, one can prove:

Theorem 2.4 (Morse)**.** *Every smooth manifold M admits a Morse function.*

Now, non-degenerate critical points are particularly nice, as locally around them f admits the following description.

Lemma 2.5 (Morse)**.** *Given any nondegenerate critical point p of f, there exists an integer k and local coordinates $x_1, \dots x_n$ around $p = (0, \dots 0)$ under which we have*

$$f(x_1, \dots x_n) = f(p) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2.$$

We thus define k to be the index of the critical point p.

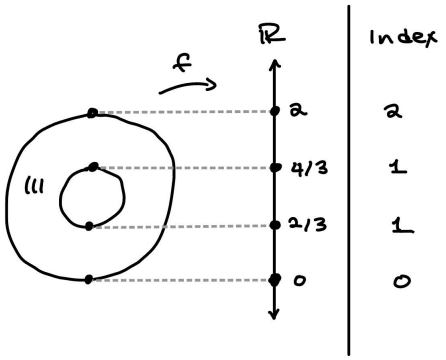
Let's now enact the Morse theory program. For any $f : M \to \mathbb{R}$ let $M^a = f^{-1}((-\infty, a])$. Here are the two central theorems:

Theorem 2.6. *[Mil63, Theorem 3.1] Consider any real numbers $a < b$ such that $f^{-1}[a, b]$ is compact and contains no critical points of f. Then M^a and M^b are diffeomorphic, and there exists a deformation retraction $M^b \to M^a$ which when pre-composed with the inclusion $M^a \hookrightarrow M^b$ gives the identity on M^a.*

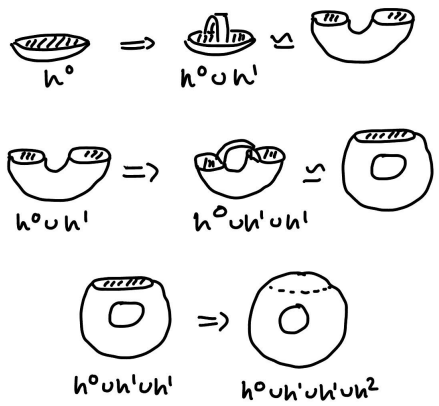
Theorem 2.7. *[Mil63, Theorem 3.2] Consider a nondegenerate critical point p of f of index k, and consider some $\epsilon > 0$ such that $f^{-1}[f(p) - \epsilon, f(p) + \epsilon]$ is compact and contains no critical point of f other than p. Then for all sufficiently small such ϵ, $M^{c+\epsilon}$ is homotopy equivalent to $M^{c-\epsilon}$ with a k-cell attached.*

Instead of proving these two theorems, we will give an example demonstrating how they work (one can find full proofs in the corresponding pages of [Mil63]).

Example 2.8. Consider the following Morse function f on a torus $T^2 = M$.



Note that f has four critical points, mapping to $0, 2/3, 4/3$, and 2. The critical points mapping to 0 and 2 have index 0 and 2 respectively, while the other two critical points have index 1. Consider $M^a = f^{-1}((-\infty, a])$. M^a first changes when a passes 0, upon which $M^a \cong D^2$ for any small $a > 0$. The diffeomorphism type of M^a changes next when a passes $2/3$, where M^a is given to a sphere with a thickened 1-cell attached. When a passes $4/3$, the diffeomorphism type of M^a again changes by attaching a 1-cell connecting the two circles bounding the cylinder. As a passes 2, M^a finally is capped off by a disk and becomes diffeomorphic to T^2. These changes are described by the diagram below. These thickened k-cell attachments are known as *k-handles* (denoted as h^k), and describe the diffeomorphism type of M^a. For more information, see [Gom99, Sections 4.1-4.2].



By piecing together Theorems 2.6 and 2.7, we may obtain the following.

Theorem 2.9. *[Mil63, Theorem 3.3] Consider a smooth manifold M and a Morse function f on M. Then M is homotopy equivalent to a CW complex with a k-cell for every critical point of f of index k.*

Proof. Assume f takes only nonnegative real values (the general case follows by an analogous argument). By Sard-type arguments one can perturb the Morse function f such that its critical points map to distinct values while retaining their indices. We first prove the theorem when M is compact. Since critical points are isolated (by Lemma 2.5), finitely many critical points exist; call them $c_1, \dots c_n$ in increasing order of $f(c_i)$. Let k_i be the index of c_i and let $\epsilon = \frac{1}{4}\min_{i=2}^{n} |f(c_i) - f(c_{i-1})|$. We claim by induction that for all $i \leq n$, $M^{c_i+\epsilon}$ is homotopy equivalent to a CW complex with a k_j-cell for each $j \leq i$; setting $i = n$ finishes the proof. The base case

$i = 1$ follows from Theorem 2.6. Now, one can readily prove (see [Mil63, Lemmas 3.4-3.5]) that the homotopy type of a space X with a k-cell attached depends only on the homotopy types of X and the attaching map. Hence $M^{c_i+\epsilon}$ and $M^{c_{i+1}-\epsilon}$ are homotopy equivalent by Theorem 2.6, and by Theorem 2.7 $M^{c_{i+1}+\epsilon}$ is homotopy equivalent to $M^{c_{i+1}-\epsilon} \cong M^{c_i+\epsilon}$ with a k_{i+1}-cell attached. Moreover, by Skeletal Approximation we may assume that this k_{i+1}-cell is attached to the $k_{i+1}-1$-skeleton of the CW complex homotopic to $M^{c_i+\epsilon}$, so the CW structure is retained. This finishes the induction and the proof when M is compact.

We now tackle the case where M is non-compact. Since the critical points of f are separated they are countable; hence enumerate them as $c_1, c_2, \ldots$ in increasing order of $f(c_i)$ (which we may do as we assumed Im $f \subset \mathbb{R}_{\geq 0}$). Then letting $a_i = (f(c_i) + f(c_{i+1}))/2$, we have that each M^{a_i} is homotopy equivalent to a CW complex K_i, with a k_j-cell for each $j \leq i$. Each K_i naturally includes into K_{i+1}, such that the homotopy equivalences $M^{a_i} \to K_i$ respect this inclusion. Let K be the direct limit of the K_i; then we retrieve a limit map $g : M \to K$. Since $\pi_n(K_i, M^{a_i}) = 0$ for all n, and homotopy groups commute with colimits, we have that g is a weak equivalence. Hence by Whitehead g is a homotopy equivalence, as desired. □

3 Morse Theory for Path Spaces

Consider a Riemannian manifold M. As discussed in the introduction, we would like to do Morse theory with respect to the energy functional

$$E(\gamma) = \int_0^1 \left|\frac{d\gamma}{dt}\right|^2 dt.$$

where $|d\gamma/dt|^2$ is evaluated via the Riemannian metric $\langle \cdot, \cdot \rangle$ on M. However, this setup is missing many components. First of all, for γ to even have derivatives, we should require γ to be in smooth. Notably, we'll need to give the space of smooth loops some topology, and prove that it's homotopy equivalent to ΓM. Even then, since the space of smooth loops is not a manifold, we have no good notion of a tangent space. Hence we will refine our Morse analogy by defining the tangent space to a path, the critical points of E, and the Hessian of E. Along the way we'll define the *Jacobi* field, which will both give us a characterization of when the Hessian of E is nondegenerate, and how to calculate its index at any critical point.

Before we begin, here are some notions from Riemannian geometry that we will use.

Definition 3.1. A geodesic $\gamma : [a, b] \to M$ is *minimal* if its length is less than or equal to the length of any other piecewise smooth path connecting its endpoints.

Definition 3.2. Given a curve $\omega : I \to M$, a vector field *along* ω W is a continuous assignment of vectors $W(t) \in TM_{w(t)}$. Alternatively, W is a continuous map $I \to TM$, such that if $\pi : TM \to M$ is the usual projection, then $\pi \circ W = \omega$.

Definition 3.3. Given any vector field W along a curve c, let $\frac{DW}{dt}$ denote the *covariant derivative* of W along c. This derivative is characterized by the fact that it is linear in W, satisfies a Leibniz rule with respect to multiplying W by a function, and is given by the usual covariant derivative $\nabla_{dc/dt}(W)$ when W is induced by a vector field on M (i.e. $W(t) = Y_{c(t)}$ for some vector field Y). For more details, see [Mil63, Chapter 8].

3.1 Smooth Path Spaces

Consider a connected Riemannian manifold M, and any two points $p, q \in M$. We'll define our smooth path spaces as follows.

Definition 3.4. A *piecewise smooth path* from p to q is a continuous map $\omega : [0,1] \to M$ satisfying that $\omega(0) = p$, $\omega(1) = q$, and there exists reals $0 = t_0 < t_1 < \ldots < t_k = 1$ such that $w|_{[t_{i-1},t_i]}$ is C^∞ for each i.

Define $\Omega^s(M; p, q)$ to be the set of piecewise smooth paths from p to q. Let ρ be the topological metric function arising from the Riemannian metric $\langle \cdot, \cdot \rangle$ on M (given by defining $\rho(x, y)$ as the infimum of the lengths of all smooth paths from x to y). We may put a topological metric d on $\Omega^s(M; p, q)$ by defining

$$d(\omega, \omega') = \max_{t\in[0,1]} \rho(\omega(t), \omega'(t)) + \left(\int_0^1 \left(\left| \frac{d\omega}{dt} \right| - \left| \frac{d\omega'}{dt} \right| \right)^2 dt \right)^{1/2}.$$

Here the second term ensures our energy function will be continuous. Now, our topology (as defined in Math 231b) on the space $\Omega(M; p, q)$ of paths from p to q was given by the compact-open topology. This topology is equivalent to the one induced by the metric

$$d'(\omega, \omega') = \max_{t\in[0,1]} \rho(\omega(t), \omega'(t)).$$

Since $d'(\omega, \omega') \leq d(\omega, \omega')$ for any ω, ω', we have that the natural inclusion $\Omega^s(M; p, q) \to \Omega(M; p, q)$ is continuous. In fact, we have the following theorem:

Theorem 3.5. *[Mil63, Theorem 17.1] The inclusion $\Omega^s(M; p, q) \to \Omega(M; p, q)$ is a homotopy equivalence.*

Note also that if M is connected, then for any points $p, q \in M$, by composing with a path from p to q (or its inverse) we get homotopy equivalences $\Omega(M; p, p) \cong \Omega(M; p, q) \cong \Omega(M; q, q)$. Thus it will suffice to prove Theorem 1.4 for the space $\Omega^s(G; p, q)$ of piecewise smooth paths between any two given points $p, q \in G$. Hence from now on we will drop the superscript s and work exclusively with piecewise smooth paths.

3.2 Tangent Spaces and Critical Points

Throughout the rest of Section 3 abbreviate $\Omega(M; p, q)$ as Ω. Though Ω is not a smooth manifold as defined, we may still make analogies of various notions from differential topology. Let's start with the tangent space.

Definition 3.6. For any $\omega \in \Omega$, define $T\Omega_\omega$ as the vector space of piecewise smooth vector fields W along ω.

To define the notion of a critical point, let us first define the *variation of a path.*

Definition 3.7. Given any $\omega \in \Omega$, a *variation of* ω is a map $\overline{\alpha} : (-\epsilon, \epsilon) \to \Omega$ for some $\epsilon > 0$, such that:

1. $\overline{\alpha}(0) = \omega$;
2. there exists reals $0 = t_0 < t_1 < \ldots < t_k = 1$ such that the map

$$\alpha : (-\epsilon, \epsilon) \times [0, 1] \to M$$

given by $\alpha(u, t) = \overline{\alpha}(u)(t)$ is C^∞ on each region $(-\epsilon, \epsilon) \times [t_{i-1}, t_i]$ for $1 \leq i \leq k$.

Definition 3.8. Consider a function $F : \Omega \to \mathbb{R}$. Then a path ω is *critical* for F if for every variation $\overline{\alpha}$ of ω, one has that

$$\frac{d(F(\overline{\alpha}(u)))}{du}\bigg|_{u=0}$$

exists and is zero.

Remark 3.9. *With additional hypotheses on F we may define its differential $dF_\omega : T\Omega_\omega \to T\mathbb{R}_{F\omega}$ for any $\omega \in \Omega$. We will not delve into more analytic details here.*

This definition of the critical path is rather clunky. It will turn out though, that when F is the energy functional $E(\omega) = \int_0^1 |d\omega/dt|^2 dt$, the critical paths will nicely correspond to *geodesics*. To prove this correspondence, we'll introduce the following formula.

Theorem 3.10. *(First Variation Formula) Consider a path $\omega \in \Omega$ and a variation $\overline{\alpha} : (-\epsilon, \epsilon) \to \Omega$ of ω. Define the vector fields W_t, V_t, A_t as:*

$$W_t = \frac{\partial \alpha}{\partial u}(0,t);$$
$$V_t = \frac{d\omega}{dt};$$
$$A_t = \frac{D}{dt}\frac{d\omega}{dt}$$

Moreover, for any vector field X_t along a curve, define

$$\Delta_t X = \lim_{s \to t^+} X_s - \lim_{s \to t^-} X_s.$$

Note that if X_t is piecewise smooth then $\Delta_t X$ is nonzero for finitely many t. Then we have that

$$\frac{1}{2}\frac{dE(\overline{\alpha}(u))}{du}\bigg|_{u=0} = -\sum_t \langle W_t, \Delta_t V\rangle - \int_0^1 \langle W_t, A_t\rangle dt.$$

Notably, $\frac{dE(\overline{\alpha}(u))}{du}\big|_{u=0}$ always exists, and only depends on W_t and ω. Using this Variation Formula, we can determine the critical paths of E.

Theorem 3.11. *[Mil63, Corollary 12.3] The critical paths of E are precisely the minimal geodesics connecting p and q.*

Proof. If ω is a geodesic, then definitionally A_t and $\Delta_t V$ are zero everywhere, so ω is a critical path of E. Now consider a critical path ω of E, smooth everywhere except at $t \in \{t_0 = 0, t_1, \dots t_k = 1\}$. Consider any smooth function $f(t) : [0,1] \to \mathbb{R}_{\geq 0}$ vanishing only at the t_i, and consider a variation $\overline{\alpha}$ of ω with corresponding vector field $W_t = f(t)A_t$ (some such variation clearly exists). Then

$$\frac{1}{2}\frac{dE(\overline{\alpha}(u))}{du}\bigg|_{u=0} = -\int_0^1 f(t)\langle A_t, A_t\rangle dt$$

is zero iff $A(t) = 0$ for all t, implying that ω must be a geodesic on each $[t_{i-1}, t_i]$. Now, pick another variation $\overline{\alpha}'$ with corresponding vector field $W_t' = \Delta_t V$. Then

$$\frac{1}{2}\frac{dE(\overline{\alpha}'(u))}{du}\bigg|_{u=0} = -\sum_t \langle \Delta_t V, \Delta_t V\rangle$$

is zero iff all $\Delta_t V = 0$, which implies ω is C^1 everywhere. Then uniqueness theorems for differential equations (e.g. [Mil63, Theorem 10.2]) show ω is C^∞ and hence a geodesic, as desired. □

3.3 The Energy Hessian

Let's now analogize the Hessian. We'll start by defining the Hessian for an arbitrary $F : \Omega \to \mathbb{R}$.

Definition 3.12. Consider a function $F : \Omega \to \mathbb{R}$, a critical path ω of F, and any two vector fields $W_1, W_2 \in T\Omega_\omega$. Pick a map $\alpha : U \times [0,1] \to M$, where U is a neighborhood of $(0,0) \in \mathbb{R}^2$, such that:

1. for some reals $0 = t_0 < \ldots < t_k = 1$, α is C^∞ on $U \times [t_{i-1}, t_i]$ for all $1 \leq i \leq k$;
2. for any t, we have

$$\alpha(0,0,t) = \omega(t), \qquad \frac{\partial \alpha}{\partial u_1}(0,0,t) = W_1(t), \qquad \frac{\partial \alpha}{\partial u_2}(0,0,t) = W_2(t).$$

Let $\overline{\alpha}(u_1, u_2) = \alpha(u_1, u_2, \cdot)$. Then define the *Hessian* H_F as

$$H_F(W_1, W_2) := \left.\frac{\partial^2 F(\overline{\alpha}(u_1, u_2))}{\partial u_1 \partial u_2}\right|_{(0,0)}$$

This notion is *a priori* not well-defined; why does it not depend on the $\overline{\alpha}$ chosen? Nevertheless, the following formula will show that the Hessian of the energy functional behaves nicely.

Theorem 3.13. *(Second Variation Formula) Retain the notation from Theorem 3.10. Consider a geodesic $\gamma \in \Omega$. Let $V_t = \frac{d\gamma}{dt}$, and let $\overline{\alpha} : U \to \Omega$ be a 2-parameter variation of γ (i.e. U is a neighborhood of $(0,0) \in \mathbb{R}^2$), with variation vector fields (for $i \in \{1,2\}$)*

$$W_i = \frac{\partial \overline{\alpha}}{\partial u_i}(0,0).$$

Then

$$\frac{1}{2} H_E(W_1, W_2) = -\sum_t \left\langle W_2(t), \Delta_t \frac{DW_1}{dt} \right\rangle - \int_0^1 \left\langle W_2, \frac{D^2 W_1}{dt^2} + \mathcal{R}(V, W_1)V \right\rangle dt$$

where $\mathcal{R}$ denotes the Riemann curvature tensor.

Notably, Theorem 3.13 implies H_E is bilinear on $T\Omega_\gamma$. Moreover, since

$$\frac{\partial^2 E(\overline{\alpha})}{\partial u_1 \partial u_2} = \frac{\partial^2 E(\overline{\alpha})}{\partial u_2 \partial u_1},$$

we have that H_E is symmetric.

Now, in our Morse theoretic setup, it was important that at the critical points of the Morse function, the Hessian was nondegenerate. To measure the nondegeneracy of H_E and calculate its index, we will introduce the *Jacobi field*.

Definition 3.14. Consider a geodesic γ, and let $V = \frac{d\gamma}{dt}$. A vector field J along γ is a *Jacobi field* if

$$\frac{D^2 J}{dt^2} + \mathcal{R}(V, J)V = 0.$$

Note that upon choosing orthonormal parallel vector fields $P_1, \dots P_n$ along γ, and letting $J(t) = \sum_i f^i(t)P_i(t)$, $a^i_j = \langle \mathcal{R}(V, P_j)V, P_j\rangle$, the Jacobi field equation becomes

$$\frac{d^2 f^i}{dt^2} + \sum_j a^i_j(t) f^j(t) = 0.$$

This is a second order linear differential equation, and hence has $2n$ linearly independent C^∞ solutions (each of which may be defined throughout γ). Notably, any Jacobi vector field J is determined by $J(0)$ and $\frac{DJ}{dt}(0)$.

Definition 3.15. Consider any geodesic $\gamma : [a, b] \to M$. Then any two distinct $a', b' \in [a, b]$ are *conjugate parameters* of γ if there exists a nonzero Jacobi field along γ vanishing at a' and b'. The *multiplicity* of the pair (a', b') as conjugate parameters is the dimension of the vector space of Jacobi fields vanishing at a' and b'.

Now, let the *null space* of H_E be the vector space of vector fields V such that $H_E(V, W) = 0$ for all vector fields W along γ, and let the *nullity* of H_E be the dimension of the null space. The following theorem will describe the null space of H_E in terms of Jacobi fields.

Theorem 3.16. *Consider a geodesic γ. A vector field $W \in T\Omega_\gamma$ lies in the null space of H_E iff W is a Jacobi field.*

Corollary 3.17. *For any geodesic $\gamma : [0, 1] \to M$, H_E is degenerate iff 0 and 1 are conjugate parameters of γ, and the nullity of H_E is the multiplicity of $(0, 1)$. Notably, the nullity of H_E at any geodesic is always finite.*

Proof of Theorem 3.16. Suppose J is a Jacobi field. Then by the Second Variation Formula, we have for any vector field W that

$$\frac{1}{2}H_E(J, W) = -\sum_t \langle W_2(t), 0\rangle - \int_0^1 \langle W_2, 0\rangle dt = 0,$$

as desired. Now suppose W_1 is in the nullity of H_E, and consider reals $0 = t_0 < t_1 < \ldots < t_k = 1$ such that W_1 is smooth on each $[t_{i-1}, t_i]$. Similarly to the proof of Theorem 3.11, we'll plug in certain values of W_2 to prove W_1 has the desired form. Indeed, let $V = \frac{d\gamma}{dt}$. Consider a smooth $f : [0, 1] \to \mathbb{R}_{\geq 0}$ vanishing only at the t_i, and let

$$W_2(t) = f(t)\left(\frac{D^2 W_1}{dt^2} + \mathcal{R}(V, W_1)V\right)_t$$

Then again by the Second Variation Formula,

$$\frac{1}{2}H_E(W_1, W_2) = -\int_0^1 f(t)\left|\frac{D^2 W_1}{dt^2} + \mathcal{R}(V, W_1)V\right|^2 dt.$$

For this to be zero, we must have that W_1 is a Jacobi field on each $[t_{i-1}, t_i]$. Now, we have that

$$\frac{1}{2}H_E\left(W_1, \Delta_t \frac{DW_1}{dt}\right) = -\sum_{i=1}^{k-1}\left|\left(\Delta_t \frac{DW_1}{dt}\right)(t_i)\right|^2,$$

which can only be zero if $\frac{DW_1}{dt}$ is continuous. Hence so is W_1; but since every Jacobi field J is determined by the value of J and $\frac{DJ}{dt}$ at any of its endpoints, we find that W_1 must be a Jacobi field along all of γ, as desired. □

In fact, we can also use Jacobi fields to describe the index of H_E at any geodesic γ (i.e. the dimension of the maximal subspace of $T\Gamma_\gamma$ on which H_E is negative definite)! This is given by the following theorem (which we will not prove; see [Mil63, Chapter 15] for details).

Theorem 3.18. *Consider any geodesic γ. The index of H_E at γ is the number of reals $t \in (0,1)$ such that 0 and t are conjugate with respect to γ, counted with multiplicity. Moreover, this index is always finite.*

Here is one way of constructing Jacobi fields. Call a variation $\overline{\alpha} : (-\epsilon, \epsilon) \to \Omega$ of a geodesic γ a *variation through geodesics* if $\overline{\alpha}(u)$ is a geodesic for all u.

Theorem 3.19. *[Mil63, Lemma 14.5] If $\overline{\alpha}$ is a variation of γ through geodesics, then $W(t) = \frac{\partial \alpha}{\partial u}(0,t)$ is a Jacobi field along γ.*

Proof. Since α is a variation through geodesics, we have $\frac{D}{dt}\frac{d\alpha}{dt} = 0$. Thus

$$\begin{aligned}\frac{D^2}{dt^2}\frac{d\alpha}{du} + \mathcal{R}\left(\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial u}\right)\frac{\partial\alpha}{\partial t} &= \frac{D}{dt}\frac{D}{du}\frac{d\alpha}{dt} + \mathcal{R}\left(\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial u}\right)\frac{\partial\alpha}{\partial t} \\ &= \frac{D}{du}\frac{D}{dt}\frac{\partial\alpha}{\partial t} = 0,\end{aligned}$$

(where we use the definition of the Riemann tensor in the first line [Mil63, Lemma 9.2] and the fact that the Levi-Civita connection is symmetric in the second line [Mil63, Lemma 8.9]), as desired. □

In fact, with some analysis, one may show that all Jacobi fields are of this form!

Theorem 3.20. *[Mil63, Lemma 14.6] Every Jacobi field along a geodesic γ may be obtained via a variation of γ along geodesics.*

4 Finite-Dimensional Approximations of Path Spaces

In section 3 we created analogies for the notions of critical points and Hessians for the energy functional on path spaces. However, since our path spaces are not finite-dimensional, we still cannot apply Theorem 2.9. In order to wield the full force of the Morse theory setup, we will now construct a finite-dimensional approximation of our path space, which will retain enough properties about the energy functional to apply Theorem 2.9.

Throughout this section consider a *complete* Riemannian manifold M and two points $p, q \in M$. Again abbreviate $\Omega = \Omega(M; p, q)$. Recall that M inherits a metric function $\rho(p,q)$ from its Riemannian metric, by letting $\rho(p,q)$ be the infimum of the lengths of all paths connecting p and q. For this construction we'll need the following lemma.

Lemma 4.1. *[Mil63, Corollary 10.11] Given a compact $K \subset M$, there exists some $\epsilon > 0$ such that any two points in K separated by a distance less than ϵ are connected by a unique (minimal) geodesic of length $< \epsilon$, which depends continuously on its endpoints.*

Now, to begin our setup, recall that the energy functional $E : \Omega \to \mathbb{R}_{\geq 0}$ is continuous; let $\Omega^c = E^{-1}([0,c]) \subset \Omega$ and $\operatorname{Int}\Omega^c = E^{-1}([0,c))$. Moreover, for any sequence of reals $0 = t_0 < t_1 < \ldots < t_k = 1$, let $\Omega(t_0, t_1, \ldots t_k)$ be the subspace of paths $\omega \in \Omega$ satisfying that $\omega|_{[t_{i-1},t_i]}$ is a geodesic for all i. Finally, define

$$\begin{aligned}\Omega(t_0, \ldots t_k)^c &= \Omega^c \cap \Omega(t_0, \ldots t_k); \\ \operatorname{Int}\Omega(t_0, \ldots t_k)^c &= (\operatorname{Int}\Omega^c) \cap \Omega(t_0, \ldots t_k).\end{aligned}$$

We first claim that for very fine subdivisions $t_0, \ldots t_k$, $\operatorname{Int}\Omega(t_0, \ldots t_k)^c$ is in fact a smooth manifold!

Theorem 4.2. *[Mil63, Lemma 16.1] Let M be a complete Riemannian manifold, and $c > 0$ a real such that $\Omega^c \neq \emptyset$. Then there exists some $\delta > 0$, such that for all sequences $0 = t_0 < \ldots < t_k = 1$ where $t_i - t_{i-1} < \epsilon$ for all i, we may give $\operatorname{Int}\Omega(t_0, \ldots t_k)^c$ the structure of a smooth finite dimensional manifold.*

Proof. For any (piecewise smooth) path ω, let $L_a^b(\omega)$ denote the arc-length of ω from $t = a$ to $t = b$. Let $S = \{x \in M : \rho(x, p) \leq \sqrt{c}\}$; since $L_0^1(\omega)^2 \leq E(\omega) \leq c$, every path $\omega \in \Omega^c$ lies entirely within S. Notably, by Lemma 4.1, there exists some $\epsilon > 0$ such that if $x, y \in S$ and $\rho(x, y) < \epsilon$, then there exists a unique minimal geodesic of length $< \epsilon$ connecting x and y (which depends continuously on x and y).

Now, consider any sequence $0 = t_0 < \ldots < t_k = 1$ such that $t_i - t_{i-1} < \delta = \epsilon^2/c$ for all i. For any broken geodesic $\omega \in \Omega(t_0, \ldots t_k)^c$, we have

$$(L_{t_{i-1}}^{t_i}\omega)^2 = (t_i - t_{i-1})(E_{t_{i-1}}^{t_i}\omega) \leq (t_i - t_{i-1})E(\omega) \leq (t_i - t_{i-1})c < \epsilon^2. \tag{1}$$

So $\omega|_{[t_i, t_{i-1}]}$ is uniquely and continuously determined by its two end points. Thus, we find that the map

$$\begin{aligned} \operatorname{Int}\Omega(t_0, \ldots t_k)^c &\to M \times M \times \ldots \times M \\ \omega &\mapsto (\omega(t_1), \omega(t_2), \ldots \omega(t_{k-1})) \end{aligned}$$

defines a homeomorphism from $\operatorname{Int}\Omega(t_0, \ldots t_k)^c$ to an open subset of the $(k-1)$-fold product $M^{\times(k-1)}$. Thus, by identifying the smooth structure on $\operatorname{Int}\Omega(t_0, \ldots t_k)^c$ with that of this open set, we have made $\operatorname{Int}\Omega(t_0, \ldots t_k)^c$ into a $(k-1)$-dimensional manifold, as desired. □

For the rest of this section let $B := \operatorname{Int}\Omega(t_0, \ldots t_k)^c$. Define

$$E' : B \to \mathbb{R}$$

to be the restriction of the energy functional $E : \Omega \to \mathbb{R}$ to B, and again let $B^a = (E')^{-1}([0, a])$.

Theorem 4.3. *[Mil63, Theorem 16.2] For every $a \leq c$, we have that B^a is a deformation retract of Ω^a. Moreover:*

1. *$E' : B \to \mathbb{R}$ is smooth;*
2. *the critical points of E' are the critical paths (i.e. geodesics) of E in $\operatorname{Int}\Omega^c$;*
3. *the index (resp. nullity) of $H_{E'}$ at any such critical point γ is equal to the index (resp. nullity) of H_E at γ.*

For the proof of this theorem we will need the following lemma (which we will not prove). Consider a geodesic $\gamma \in \Omega^c$, and let $T\Omega_\gamma(t_0, t_1, \ldots t_k) \subset T\Omega_\gamma$ be the subspace of vector fields along γ which vanish at $t = 0, 1$ and restrict to a Jacobi field along each $[t_{i-1}, t_i]$.

Lemma 4.4. *[Mil63, Lemma 15.4] The index (resp. nullity) of H_E is equal to the index (resp. nullity) of H_E restricted to $T\Omega_\gamma(t_0, t_1, \ldots t_k)$.*

Proof of Theorem 4.3. First, note that for any broken geodesic $\omega \in B$, we have that

$$\begin{aligned} E'(\omega) &= \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left|\frac{d\omega}{dt}\right|^2 dt \\ &= \sum_{i=1}^{k} \frac{\rho(\omega(t_{i-1}), \omega(t_i))^2}{t_i - t_{i-1}}. \end{aligned}$$

Since any two $x, y \in S$ with $\rho(x, y) < \epsilon$ are connected by a minimal geodesic, and every pair $(\omega(t_{i-1}), \omega(t_i))$ satisfies this condition, it is well-known that $\rho(\omega(t_{i-1}), \omega(t_i))$ depends smoothly on $\omega(t_{i-1})$ and $\omega(t_i))$. Thus $E' : B \to \mathbb{R}$ is smooth.

We now construct a retraction $r : \operatorname{Int} \Omega^c \to B$ as follows. For any $1 \leq i \leq k$, we know by the completeness of M and Hopf-Rinow that $\omega(t_{i-1})$ and $\omega(t_i)$ are connected by a minimal geodesic; by an analogous calculation to Equation (1), this geodesic is unique and has length $< \epsilon$. Thus let $r(\omega) \in B$ be the unique broken geodesic such that $r(\omega)|_{[t_{i-1}, t_i]}$ is a geodesic of length $< \epsilon$ connecting $\omega(t_{i-1})$ and $\omega(t_i)$. We may generalize this retraction into a one-parameter family of maps

$$r_u : \operatorname{Int} \Omega^c \to \operatorname{Int} \Omega^c$$

where $u \in [0, 1]$, given as follows. Suppose $u \in [t_{m-1}, t_m]$ for some m. Then

1. for $i < m$, let $r_u(\omega)|_{[t_{i-1}, t_i]} = r(\omega)|_{[t_{i-1}, t_i]}$;
2. let $r_u(\omega)|_{[t_{m-1}, u}$ be the minimal geodesic connecting $\omega(t_{i-1})$ to $\omega(u)$;
3. let $r_u(\omega)|_{[u,1]} = \omega|_{[u,1]}$.

Again by Equation (1) we have that $r_u(\omega)|_{[t_{m-1}, u}$ is uniquely defined. Now, r_0 is the identity on $\operatorname{Int} \Omega^c$, and $r_1 = r$. Moreover, one may readily verify that r_u is continuous. Thus r_u gives a deformation retraction between r and the identity on Ω^c, as desired. An analogous proof gives a deformation retract from Ω^a onto B^a.

Now, since geodesics are broken geodesics, every critical path of E is a critical point of E' in B. By applying the First Variation Formula and reiterating the proof of Theorem 3.11, we find that every critical point of E' must in fact be a geodesic, as desired.

Finally, consider any geodesic γ in B. Every vector in TB_γ is the velocity vector at time 0 of some path $\overline{\alpha} : (-\delta, \delta) \to B$ sending 0 to γ. This path is clearly a variation of γ, and thus corresponds to a Jacobi field in $T\Omega_\gamma(t_0, \dots t_k)$. One may readily verify that this produces an identification $TB_\gamma \cong T\Omega_\gamma(t_0, \dots t_k)$, which also identifies $H_{E'}$ on TB_γ with the restriction of H_E to $T\Omega_\gamma(t_0, \dots t_k)$. Hence by Lemma 4.4, $H_{E'}$ and H_E have the same index (resp. nullity) at any geodesic γ, as desired. □

Combining Theorem 4.3 and Theorem 2.9, we retrieve:

Theorem 4.5. *[Mil63, Theorem 16.3] Consider a complete Riemannian manifold M, a real $c > 0$, and two points $p, q \in M$ such that for any geodesic $\gamma \in \Omega(M; p, q)$ of length $\leq \sqrt{c}$, 0 and 1 are not conjugate parameters. Then Ω^c is homotopy equivalent to a finite CW complex with a k-cell for each geodesic in Ω^c at which H_E has index k.*

Now, a theorem of Milnor says that Ω is homotopy equivalent to a CW complex. Thus, by applying a proof analogous to that of Theorem 2.9, we may obtain the following theorem.

Theorem 4.6. *Consider a complete Riemannian manifold M and two points $p, q \in M$, such that for any geodesic $\gamma \in \Omega(M; p, q)$, 0 and 1 are not conjugate parameters. Then $\Omega(M; p, q)$ is homotopy equivalent to a (countable) CW complex with a k-cell for every geodesic connecting p to q with index k under H_E.*

Of course, this theorem would not be useful if no such pairs of "non-conjugate points" exists! Thankfully for geodesically complete Riemannian manifolds M, conjugate parameters have the following nice description.

Definition 4.7. Consider any $p \in M$. Recall that for every $v \in T_pM$, there exists a unique geodesic $\gamma : \mathbb{R} \to M$ with $(d\gamma/dt)(0) = v$ (see [Mil63, Lemma 10.3]). Thus for every $v \in T_pM$, define

$$\exp_p : T_pM \to M$$
$$\exp_p(v) := \gamma(1)$$

Theorem 4.8. *[Mil63, Theorem 18.1] Consider any $p \in M$ and $v \in T_pM$. Then 0 and 1 are conjugate parameters of γ_v iff $\exp_p$ is critical at v.*

Corollary 4.9. *[Mil63, Corollary 18.2] Consider any $p \in M$. Then for almost all $q \in M$, one has that for every geodesic $\gamma \in \Omega(M; p, q)$, 0 and 1 are not conjugate parameters.*

Proof. Apply Sard's theorem to Theorem 4.8. □

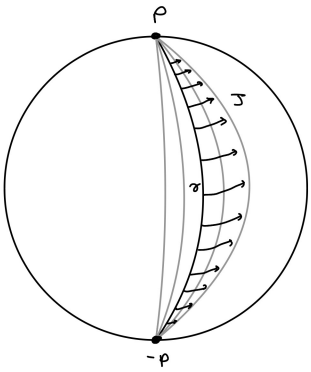
4.1 Examples

Let's investigate some examples of Theorem 4.6 in action.

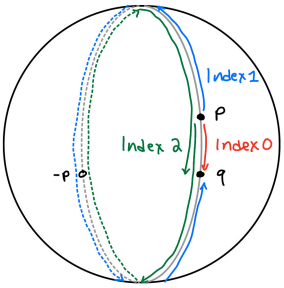
Example 4.10. Equip the n-torus $T^n = (S^1)^n = \mathbb{R}^n/\mathbb{Z}^n$ with the flat metric inherited from $\mathbb{R}^n$. The geodesics of T^n are the images of straight lines under the quotient map $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$. Notably, given any two distinct points $p, q \in T^n$, the set of geodesics connecting p and q is the set of images of lines connecting any lift of p to any lift of q; since the lifts of any point in T^n form a lattice in $\mathbb{R}^n$, the number of such geodesics must be countable. Notably, no geodesic has a variation along geodesics, and so no two distinct reals are conjugate parameters under any geodesic. Hence all geodesics in any $\Omega(T^n; p, q)$ have index 0, and so $\Omega(T^n)$ is homotopy equivalent to a discrete set. But we knew this as $\Omega(T^n) = \Omega(K(\mathbb{Z}^n, 1)) = K(\mathbb{Z}^n, 0)$ is just $\mathbb{Z}^n$ equipped with the discrete topology.

Example 4.11. Equip S^n with the round metric; then the geodesics of S^n are the great circles (i.e. the intersections of dimension 2 subspaces of $\mathbb{R}^{n+1}$ with S^n). Given any $p, q \in S^n$ and geodesic $\gamma \in \Omega(S^n; p, q)$, if p and q are not antipodal along S^n, then p and q lie on a unique great circle, and hence only countably many geodesics connect p and q. Notably γ admits no variations along geodesics, so p and q are not conjugate.

Meanwhile, suppose p and q are antipodal, and consider a minimal geodesic γ connecting p and q. Any such geodesic is determined by $\gamma(1/2)$, which must lie on $S^n \cap p^\perp \cong S^{n-1}$. Rotating γ in the directions specified by the vectors in $T_{\gamma(1/2)}S^{n-1}$ specifies a $(n-1)$-dimensional space of Jacobi fields. In fact this must be the full space of Jacobi fields along γ vanishing at p and q, as such a Jacobi field J is uniquely specified by $\frac{dJ}{dt}(0)$, and if $\frac{dJ}{dt}(0)$ is parallel to $\frac{d\gamma}{dt}$ then J corresponds to a variation along geodesic which "speeds up" the geodesic γ. But this would imply $J(1) \neq 0$, a contradiction; this leaves an $(n-1)$-dimensional space of possible Jacobi fields, all of which is realized. So for any minimal geodesic connecting antipodes, 0 and 1 are conjugate parameters with multiplicity $n-1$. Similarly, for any geodesic running an angle of $k\pi$ around a great circle (for $k \in \mathbb{Z}$), 0 and 1 are conjugate parameters with multiplicity $|k|(n-1)$. The diagram below depicts the unique nonzero Jacobi field (up to scaling) for a minimal geodesic γ connecting antipodes on S^2.



Now, pick any two points $p, q \in S^n$ which are not antipodes of each other. For every $k \in \mathbb{Z}_{\geq 0}$, there exists exactly one geodesic $\gamma \in \Omega(S^n; p, q)$ passing through either p or its antipode a combined total of k times (excluding $t = 0$). Such a geodesic thus has index $k(n-1)$ (the diagram below shows the geodesics of index $0, 1, 2$ between two generic points on S^2). Hence by Theorem 4.6, $\Omega(S^n) \cong \Omega(S^n; p, q)$ is homotopy equivalent to a CW complex with exactly one $k(n-1)$-cell for each $k \in \mathbb{Z}_{\geq 0}$, and no other cells. Notably, for $n \geq 3$ this implies $H_m(\Omega S^n)$ is $\mathbb{Z}$ if $n-1|m$ and trivial otherwise; a fact we derived in class via the Serre spectral sequence!



5 Proving the Main Theorem

Now that we've established a Morse theory setup for path spaces, a proof of Theorem 1.4 will follow after introducing some preliminaries on symmetric spaces and Lie groups.

5.1 Symmetric Spaces

Definition 5.1. A *symmetric space* is a connected Riemannian manifold M satisfying that for any $p \in M$, there exists an isometry $I_p : M \to M$ which fixes p but reverses geodesics through p (i.e. $I_p\gamma(t) = \gamma(-t)$).

Consider any geodesic $\gamma : (a, b) \to M$, and consider $c, d \in (a, b)$. A simple calculation gives that for any t, $I_{\gamma(d)} I_{\gamma(c)} \gamma(t) = \gamma(t + 2(d - c))$; hence we may infinitely extend any geodesic γ. Thus by Hopf-Rinow M must be complete [Mil63, Theorem 10.12, Corollary 10.13].

Symmetric spaces will be useful to us due to the following two theorems.

Theorem 5.2. *[Mil63, Corollary 20.4] Consider a geodesic $\gamma : \mathbb{R} \to M$ on a symmetric space M. If U, V, W are any three vector fields parallel along γ, then $\mathcal{R}(U, V)W$ is also parallel along γ.*

Proof. Consider any reals $a, b \in \mathbb{R}$, and any other vector field X parallel along γ. One may verify via an elementary calculation that $T = I_{\gamma((a+b)/2)} I_{\gamma(a)}$ is an isometry and preserves parallel vector fields [Mil63, Corollary 20.1]. Thus we have:

$$\begin{aligned}\langle \mathcal{R}(U_b, V_b)W_b, X_b\rangle &= \langle \mathcal{R}(dT_{\gamma(a)}U_a, dT_{\gamma(a)}V_a)dT_{\gamma(a)}W_a, dT_{\gamma(a)}X_a\rangle \\ &= \langle \mathcal{R}(U_a, V_a)W_a, X_a\rangle.\end{aligned}$$

So $\langle \mathcal{R}(U,V)W, X\rangle$ is constant for all X, which readily implies that $\mathcal{R}(U,V)W$ is a parallel vector field as desired [Mil63, Lemma 8.5]. □

Now, for symmetric spaces, the solutions to the Jacobi equation are nicely expressible in the following manner.

Definition 5.3. Consider a geodesic $\gamma : \mathbb{R} \to M$, and let $V = \frac{d\gamma}{dt}$ be the velocity vector field of γ. Then define the linear map $K_V : T_pM \to T_pM$ as

$$K_V(W) := \mathcal{R}(V_0, W)V_0.$$

Theorem 5.4. *[Mil63, Theorem 20.5] Fix the notation above. Let $e_1, \dots e_n$ be the positive eigenvalues of K_V. Then*

1. *the conjugate parameters to 0 are given by the reals $t = \pi k/\sqrt{e_i}$, where $k \in \mathbb{Z}$ is nonzero;*
2. *the multiplicity of any pair $(0, t)$ is given by the number of e_i such that t is an integer multiple of $\pi/\sqrt{e_i}$.*

Proof. Recall the following symmetry relation of the Riemann curvature tensor:

$$\langle \mathcal{R}(V, W)V', W'\rangle = \langle \mathcal{R}(V', W')V, W\rangle.$$

Setting $V' = V$ gives us that K_V is self-adjoint. Hence by the spectral theorem we may consider orthonormal vectors $U_1, \dots U_m \in M_p$, with eigenvalues $e_1, \dots e_m$ with respect to K_V, which we may then extend to vector fields along γ via parallel translation. Since M is symmetric, the equation

$$\mathcal{R}(V, U_i)V = e_i U$$

remains true along γ.

Now consider any vector field W along γ. We may express W uniquely as

$$W(t) = \sum_{i=1}^{m} w_i(t)U_i(t).$$

Now, the Jacobi equation states

$$\begin{aligned}\frac{D^2W}{dt^2} + K_V(W) &= 0 \\ \sum_{i=1}^{n}\left(\frac{d^2w_i}{dt^2}U_i + e_i w_i U_i\right) &= 0\end{aligned}$$

Hence we must have that

$$\frac{d^2w_i}{dt^2} + e_i w_i = 0$$

for each i. The solutions to this equation are linear combinations of $e^{\pm t\sqrt{-e_i}}$ if $e_i \neq 0$, and linear functions in t if $e_i = 0$. The only such solutions which vanish at $t = 0$ and another nonzero value of t satisfy that $e_i > 0$. In that case each such solution contributes a zero to each nonzero multiple of $\pi/\sqrt{e_i}$, as desired. □

5.2 Compact Lie Groups

Now, following Theorem 1.4, consider a compact Lie group G. Let $L_\tau : \sigma \to \tau\sigma$ and $R_\tau : \sigma \to \sigma\tau$ denote the (smooth) left and right multiplication maps by any $\tau \in G$. We'll first show that G admits a left and right invariant Riemannian metric. Indeed, start with any Riemannian metric $\langle \cdot, \cdot \rangle'$ on G, and let μ denote the Haar measure on G. Then, given any two vectors $V, W \in T_pG$, define

$$\langle V, W \rangle = \int_{G \times G} \langle d(L_\sigma)_{\tau p} \circ d(R_\tau)_p(V), d(L_\sigma)_{\tau p} \circ d(R_\tau)_p(W) \rangle' \, d\mu(\sigma) d\mu(\tau).$$

One may readily verify that $\langle \cdot, \cdot \rangle$ is left and right invariant. Henceforth equip G with $\langle \cdot, \cdot \rangle$. We claim that with this metric, G in fact becomes a symmetric space.

Theorem 5.5. *The Riemannian manifold $(G, \langle \cdot, \cdot \rangle)$ is a symmetric space, where for any $\tau \in G$, the isometry $I_\tau : G \to G$ is given by $\sigma \to \tau\sigma^{-1}\tau$.*

Proof. Evidently I_τ fixes τ for all $\tau \in G$. Now, we first check the result for $\tau = e$. In that case $I_e(\sigma) = \sigma^{-1}$, so I_e acts by negative the identity on T_eG. So I_e is an isometry at e. Moreover, for any $\sigma \in G$, we have $I_e L_\sigma = R_{\sigma^{-1}} I_e$, and thus $d(I_e)_\sigma \circ d(L_\sigma)_e = d(R_{\sigma^{-1}})_e \circ d(I_e)_e$; since L_σ and $R_{\sigma^{-1}}$ are both isometries, $d(I_e)_\sigma : T_\sigma G \to T_{\sigma^{-1}} G$ is also an isometry. So I_e is an isometry everywhere; since it acts by -1 on T_eG, it reverses the geodesics passing through e.

Now, for any $\tau \in G$, we have $I_\tau = R_\tau I_e R_\tau^{-1}$. Thus we readily retrieve that I_τ is an isometry and reverses geodesics through τ, as desired. □

Note that geodesics in G have the following nice form.

Definition 5.6. A *one-parameter subgroup* of G is a C^∞ homomorphism from $\mathbb{R}$ to G.

Lemma 5.7. *[Mil63, Lemma 21.2] The geodesics $\gamma : \mathbb{R} \to G$ with $\gamma(0) = e$ are precisely the one-parameter subgroups of G.*

Now, say that a vector field X on G is *left invariant* if for all $a, b \in G$, $d(L_a)_b X_b = X_{ab}$. Recall that the Lie algebra $\mathfrak{g}$ associated to G is the space of left-invariant vector fields on G equipped with the Lie bracket. Notably, every left-invariant vector field is determined by its vector at e, so we may identify $\mathfrak{g} \cong T_eG$. The following identities of left-invariant vector fields will prove useful soon.

Theorem 5.8. *[Mil63, Theorem 21.3] Given any left-invariant vector fields X, Y, Z on G, one has that*

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle$$

and

$$\mathcal{R}(X, Y)Z = \frac{1}{4}[[X, Y], Z].$$

5.3 The Main Theorem

We now finally prove Theorem 1.4.

Proof of Theorem 1.4. By Corollary 4.9, we may pick a $p \in G$ such that no geodesics in $\Omega = \Omega(G; e, p)$ have 0 and 1 as conjugate parameters. Note that by the discussion in Section 3.1, it suffices to show that Ω has the desired CW decomposition. By Theorem 4.6, it suffices to show that any geodesic in Ω always has even index; we do this now.

Consider any geodesic $\gamma \in \Omega$ (which we may infinitely extend as G is a symmetric space), and let $V = d\gamma/dt$ be its velocity vector field. Since $\gamma(0) = e$, γ is a one-parameter subgroup of G, and so V is the left-invariant vector field determined by V_0. Now, recall the linear map

$$K_V : T_eM \to T_eM$$
$$K_V(W) := \mathcal{R}(V, W)V = \frac{1}{4}[[V, W], V].$$

Here the last line is by identifying $T_eM \cong \mathfrak{g}$ and applying Theorem 5.8. Consider the adjoint homomorphism

$$\mathrm{Ad}_V : \mathfrak{g} \to \mathfrak{g}$$
$$\mathrm{Ad}_V(W) := [V, W].$$

Then

$$K_V = -\frac{1}{4}(\mathrm{Ad}_V)^2,$$

and by Theorem 5.8 we have that K_V is skew-symmetric. Well-known spectral theory thus demonstrates that under some orthonormal basis of $\mathfrak{g}$, the matrix for Ad_V is block-diagonal with blocks of the form [0] or

$$\begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix},$$

where $\lambda \in \mathbb{R}$. Since

$$\begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}^2 = \begin{bmatrix} -\lambda^2 & 0 \\ 0 & -\lambda^2 \end{bmatrix},$$

we have that the negative eigenvalues of Ad_V^2 come in pairs. Thus by Theorem 5.4, we find that the index of λ is even, as desired. □

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